

Geodesic strikes for composite, basket, Asian, and spread options

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Abstract

We discuss simple methodologies for the selection of *most relevant* or *effective* strikes for the assessment of appropriate implied volatilities used for the valuation of composite, basket, Asian, and spread options following the spirit of *geodesic strikes* [ABOBF02].

Introduction

In the context of vanilla, or near-vanilla, derivatives trading, we often also encounter *composite*, *basket*, *Asian*, and *spread* options. Whilst their respective payoff rules do in principle warrant treating them as genuinely exotic options, it is often desirable to use a simple approximation for the sake of tractability. A popular approach is to use a relatively simple valuation methodology based on multivariate geometric Brownian motion, the well-known *Black-Scholes-Merton* framework, and to find a suitable implied volatility (or term structure thereof) for each underlying selected by the concept of a *most relevant*, or *effective*, strike for each observation date. For a plain vanilla option, clearly, the volatility must be taken from the implied volatility surface at the option's expiry date and at the option's strike. For composite, basket, Asian, and spread options, the most suitable strike for the looking up of implied volatility is not necessarily as obvious. In this document, we suggest a systematic procedure that addresses this question.

Composite, Basket, Asian, and spread options

A *composite* option is a contract of European style which pays at a future payment date T_{pay} the payoff

$$(\theta \cdot [X(T) \cdot Y(T) - K])_+, \quad (2.1)$$

with $\theta = \pm 1$ for calls and puts, based on two underlyings X and Y observed on the expiry date $T \leq T_{\text{pay}}$.

In practice, it is not uncommon for X to be the market price of a standard investment asset such as an equity share, denominated in its domestic currency DOM, and for Y to be an FX rate that converts the final investment asset value to a target currency TAR. As a consequence of the natural direction of the FX rate, i.e., DOMTAR, meaning the value of one DOM currency unit expressed in units of TAR, the valuation of a composite option may also incur a quanto effect since the asset X and the FX rate DOMTAR cannot be martingales in the same measure, and this has to be taken into account. For the purpose of this article, however, we shall assume that all involved underlyings are martingales in the same measure. In practice, this may mean that we have to determine an effective quanto forward for the asset X in the target currency TAR by other means of approximation prior to being able to commence with our effective geodesic strike procedure. We shall return to this point at the end of section 3.

Basket options are, conventionally, derivatives with a payoff of the form

$$(\theta \cdot [\sum_i w_i X_i(T) - K])_+. \quad (2.2)$$

This is a vanilla option on a linearly weighted average of a number of underlyings, whence it is also referred to as an *arithmetic* basket option. In contrast, whilst rarely traded, there is also the *geometric* basket option

$$(\theta \cdot [\prod_i X_i^{w_i}(T) - K])_+. \quad (2.3)$$

Asian options are in the framework of multivariate geometric Brownian motion merely a special case of arithmetic basket options in that the fixings that contribute to the average are from the same underlying, but for different observation times:

$$(\theta \cdot [\sum_i w_i X(T_i) - K])_+. \quad (2.4)$$

Spread options usually simply pay according to

$$(\theta \cdot [(X(T) - Y(T)) - K])_+ \quad (2.5)$$

for two underlyings X and Y .

By allowing the subscript i on an observation index X_i to indicate a specific underlying as well as fixing time, and by additionally permitting weights to be

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negative as well as positive, it is clear that all of basket, Asian, and spread options take on the form of an arithmetic average option

$$(\theta \cdot [\sum_i w_i X_i - K])_+ . \quad (2.6)$$

Equally, it is evident that both composite and geometric basket options appear as a geometric average option

$$(\theta \cdot [\prod_i X_i^{w_i} - K])_+ . \quad (2.7)$$

Subsequently, we will therefore concentrate on these two generic cases: geometric and arithmetic average options.

Geodesic strikes

The formal derivation of the procedure for the calculation geodesic strikes in [ABOBF02] involves concepts of projection onto an effective local volatility representation of a basket process for large deviations, as well as Varadhan's geodesic theorem [Var67], and is rather technical. In this document, we shall attempt to obtain a similar result with a somewhat less rigorous, though tractable, argument, bearing in mind that the sole purpose of the exercise is to arrive at a set of suitably chosen effective strikes for the lookup of implied volatilities from the underlyings' implied volatility smiles. These volatilities are then to be used in whatever near-vanilla approximation that is chosen for the respective target product. We emphasize that it is clear that this process cannot possibly arrive at a sophisticated exotic product pricing framework. Instead, it merely is intended to give a procedure that suffices for the simplistic, but very fast, valuation of some near-vanilla products whilst preserving some sensible consistency conditions.

The starting point is that all of the involved stochastic financial observables \mathbf{X} are governed by a joint log-normal law, and that there is a critical level K for a function $f(\cdot)$ of the financial observables, identified by the fact that the payout is of the form

$$(\theta \cdot (f(\mathbf{X}) - K))_+ . \quad (3.1)$$

As is well known, without loss of generality, we can transform the vector of financial variables \mathbf{X} to a vector of independent standard Gaussian variables \mathbf{z} according to

$$X_i = \hat{X}_i \cdot e^{-\frac{1}{2}c_{ii} + \sum_j a_{ij}z_j} \quad (3.2)$$

with

$$\hat{\mathbf{X}} = \langle \mathbf{X} \rangle , \quad (3.3)$$

the matrix A being the *dispersion* or *factor loading* matrix, and the log-covariance matrix C whose elements are

$$c_{ij} = \langle \ln X_i, \ln X_j \rangle \quad (3.4)$$

relating to the dispersion matrix A according to

$$C = A \cdot A^\top . \quad (3.5)$$

The approximation of *effective geodesic strikes* is now to find a set of logarithmic shift coefficients $\boldsymbol{\xi}^*$, which relates to the effective strike for underlying $\#i$ as

$$K_i^* = \hat{X}_i \cdot e^{\xi_i^*} , \quad (3.6)$$

such that the multivariate probability density of $\boldsymbol{\xi}$ under the joint normal law with

$$\langle \boldsymbol{\xi} \cdot \boldsymbol{\xi}^\top \rangle = C \quad (3.7)$$

is maximal at $\boldsymbol{\xi} = \boldsymbol{\xi}^*$ subject to the constraint

$$f(\mathbf{K}^*) = K . \quad (3.8)$$

In other words, we seek

$$\boldsymbol{\xi}^* = \arg \max_{\boldsymbol{\xi}} \left[\psi(\boldsymbol{\xi}) \mid f(\mathbf{K}^*) = K \right] \quad (3.9)$$

with

$$\psi(\boldsymbol{\xi}) := \frac{e^{-\frac{1}{2}\boldsymbol{\xi}^\top C^{-1}\boldsymbol{\xi}}}{\sqrt{(2\pi)^n \cdot |C|}} \quad (3.10)$$

and \mathbf{K}^* being given elementwise in equation (3.6). The log-bilinear form of (3.10) allows us to simplify (3.9) to

$$\mathbf{z}^* = \arg \min_{\mathbf{z}} \left[\mathbf{z}^\top \cdot \mathbf{z} \mid f(\mathbf{K}^*) = K \right] \quad (3.11)$$

with

$$\boldsymbol{\xi}^* = A \cdot \mathbf{z}^* . \quad (3.12)$$

As we shall see below, it turns out that the effective strikes themselves depend on (implied) volatilities. This makes the task of effective geodesic strike calculation ultimately an implicit problem, since we need the effective strikes to be able to look up the implied volatilities in the first place. In practice, we resolve this by the approximation that all volatilities that show up in the effective strike formulæ are to be taken as at-the-forward implied volatilities. In this context, we recall that we mentioned at the end of the first paragraph in section 2 that, when some of the underlyings require translation into the target valuation measure, an approximation for this translation of forward and implied volatility smile has to be employed separately and prior to the invocation of the effective geodesic strike procedure. When choosing the quanto-translation procedure, it is useful to bear in mind that the subsequent geodesic strike selection is only an approximation for the sake of analytic tractability for a range of near-vanilla products,

and thus judge the required level of sophistication for the quanto translation in line with the overall level of the chain of approximations. We emphasize, however, that whilst the geodesic strike procedure is only an approximation, its purpose is to be consistent and accurate in a certain asymptotic sense, namely that of the local volatility projection for large deviations, i.e, for out-of-the-money options, and that of the geodesic distance asymptotics of [ABOBF02] and [Var67].

Geodesic strikes for geometric average options

We consider the payoff $f(\cdot)$ being given by

$$f_{\text{geometric average}}(\mathbf{X}) = \prod_i X_i^{w_i}. \quad (3.13)$$

The constraint (3.8) becomes

$$K = \hat{G} \cdot e^{\mathbf{w}^\top \cdot A \cdot \mathbf{z}^*} \quad (3.14)$$

with

$$\hat{G} := \prod_i \hat{X}_i^{w_i} \quad (3.15)$$

which we write as

$$\mathbf{w}^\top \cdot A \cdot \mathbf{z}^* - \kappa = 0 \quad (3.16)$$

with

$$\kappa := \ln(K/\hat{G}). \quad (3.17)$$

In order to minimize the L_2 -norm of \mathbf{z}^* subject to the constraint (3.16), we solve the Lagrange multiplier problem

$$\mathbf{z}^* = \arg \min_{\mathbf{z}} \left[\frac{1}{2} \cdot \mathbf{z}^\top \cdot \mathbf{z} - \lambda \cdot \left(\mathbf{w}^\top \cdot A \cdot \mathbf{z} - \kappa \right) \right]. \quad (3.18)$$

From

$$\nabla_{\mathbf{z}} \left(\frac{1}{2} \cdot \mathbf{z}^\top \cdot \mathbf{z} - \lambda \cdot \left(\mathbf{w}^\top \cdot A \cdot \mathbf{z} - \kappa \right) \right) \Big|_{\mathbf{z}=\mathbf{z}^*} = 0 \quad (3.19)$$

we have

$$\mathbf{z}^* = \lambda \cdot A^\top \cdot \mathbf{w} \quad (3.20)$$

which we substitute into (3.16) yielding

$$\lambda = \frac{\kappa}{\mathbf{w}^\top \cdot C \cdot \mathbf{w}}. \quad (3.21)$$

Upon resubstitution into (3.20), we arrive at

$$\mathbf{z}^* = \frac{\kappa \cdot A^\top \cdot \mathbf{w}}{\mathbf{w}^\top \cdot C \cdot \mathbf{w}}. \quad (3.22)$$

As for the effective geodesic strikes, equation (3.6) gives us

$$K_i^* = \hat{X}_i \cdot e^{\xi_i} \quad (3.23)$$

with

$$\xi_i := \frac{\kappa \cdot \sum_j c_{ij} w_j}{\mathbf{w}^\top \cdot C \cdot \mathbf{w}} \quad (3.24)$$

for geometric average options. For composite options as defined in (2.1), we have $w_i \equiv 1$ and

$$K_x^* = \hat{X} \cdot e^{\left(\ln\left(\frac{K}{\hat{X}\hat{Y}}\right) \cdot \frac{\sigma_x \cdot (\sigma_x + \rho_{XY} \sigma_Y)}{\sigma_x^2 + 2\sigma_x \rho_{XY} \sigma_Y + \sigma_Y^2} \right)}$$

$$K_y^* = \hat{Y} \cdot e^{\left(\ln\left(\frac{K}{\hat{X}\hat{Y}}\right) \cdot \frac{\sigma_Y \cdot (\sigma_Y + \rho_{XY} \sigma_X)}{\sigma_x^2 + 2\sigma_x \rho_{XY} \sigma_Y + \sigma_Y^2} \right)}. \quad (3.25)$$

It is worth reflecting on equation (3.25) with respect to a few benchmark cases. First, let us consider the case when $\sigma_Y \rightarrow 0$. In that case, we have

$$\lim_{\sigma_Y \rightarrow 0} K_x^* = \frac{K}{\hat{Y}} \quad (3.26)$$

which is consistent with the exact plain vanilla option we arrive at in this limit for a composite option, and we obtain of course the symmetric equivalent for $\sigma_x \rightarrow 0$. When correlation is zero, the log-moneyness of the composite option translates to log-moneyness of the underlyings as a function of the log-volatility-ratio

$$\nu := 2 \ln(\sigma_x/\sigma_Y) \quad (3.27)$$

according to the logistic functions

$$\xi_x^* = \frac{\kappa}{1 + e^{-\nu}} \quad \text{and} \quad \xi_y^* = \frac{\kappa}{1 + e^{\nu}} \quad (3.28)$$

as shown in figure 1. The graph illustrates the sym-

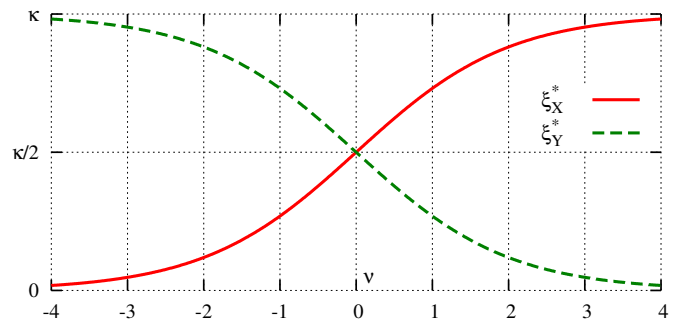


Figure 1: Underlying log-moneyness for composite options with zero correlation according to equation (3.28).

metry $\xi_x^*(\nu) + \xi_y^*(\nu) = 1$ which is intuitively appealing. Also, when volatilities are equal, the log-moneyness κ is equally distributed over both underlyings, which is again what one would intuitively expect from a sensible effective strike approximation formula. In the limit of $\rho_{XY} = 1$, we obtain

$$\xi_x^* = \frac{\kappa}{1 + e^{-\frac{\nu}{2}}} \quad \text{and} \quad \xi_y^* = \frac{\kappa}{1 + e^{\frac{\nu}{2}}} \quad (3.29)$$

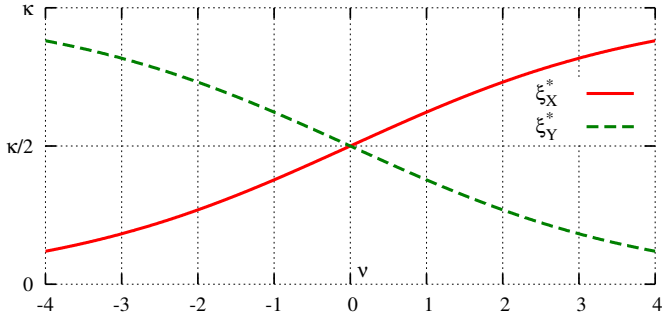


Figure 2: Underlying log-moneyness for composite options with perfect correlation according to equation (3.29).

which we display in figure 2. We see in figure 2 that with diverging volatilities, i.e., as ν diverges from 0, the underlyings' log-moneyness is less rapidly shifted from an equal split at $\nu = 0$ to just one of the underlyings, as indicated by the lesser slope near zero, when comparing with the case $\rho_{XY} = 0$ in figure 1. This behaviour is also intuitively desirable.

Finally, we consider the case of perfect correlation given by the option under consideration actually being a *square option*

$$(\theta \cdot (X^2 - K))_+ . \quad (3.30)$$

In comparison, by a standard argument of continuous replication, we can value this option based on the Taylor expansion with complete remainder for any smooth function $h(x)$ around k

$$h(x) = h(k) + h'(k)(x - k) + \int_k^x h''(z)(x - z) dz \quad (3.31)$$

whence

$$h(x) \cdot \mathbf{1}_{\{x > k\}} = h(k) \cdot \mathbf{1}_{\{x > k\}} + h'(k)(x - k)_+ + \int_k^\infty h''(z)(x - z)_+ dz . \quad (3.32)$$

Choosing $h(X) = X^2 - K$ and $k := \sqrt{K}$, and taking the expectation over X , we obtain for the call option

$$\begin{aligned} \mathbb{E} \left[(X^2 - K)_+ \right] & \quad (3.33) \\ & = 2kB(\hat{X}, k, \hat{\sigma}(k)) + 2 \int_k^\infty B(\hat{X}, k', \hat{\sigma}(k')) dk' \end{aligned}$$

wherein $\hat{\sigma}(k')$ is the implied volatility of the asset for strike k' , and $B(\hat{X}, K, \sigma)$ is the Black vanilla call option function for forward \hat{X} , strike K , and implied volatility σ . Equation (3.33) has two parts. First, there is a vanilla call option struck at $k = \sqrt{K}$ with absolute weight $2k$. Second, we have a continuum of vanilla call options for all strikes above k with density 2. For significantly out-of-the-money options, as stipulated by the large deviations asymptotics at the heart of the geodesic argument, the dominant part will be the discretely weighted call option struck at

$k = \sqrt{K}$, and the contribution of the continuous part will be largely centered near k and tail off rapidly as $k' \rightarrow \infty$ due to the rapid decay of out-of-the-money options value $B(\hat{X}, k', \hat{\sigma}(k'))$, assuming that $\hat{\sigma}(k')$ rises only moderately such as would be consistent with finite second and higher moments. Hence, while the selection of an effective implied volatility at $k = \sqrt{K}$ for the underlying does not exactly reproduce the smile dependence one can obtain from the continuous integration over all strikes to infinity (3.33), we at least capture the fact that the dominant contribution does indeed come from the implied volatility near $k = \sqrt{K}$, as intended.

Geodesic strikes for arithmetic average options

Here, we have

$$f_{\text{arithmetic average}}(\mathbf{X}) = \sum_i w_i X_i . \quad (3.34)$$

The constraint (3.8) is

$$g(\mathbf{z}^*) = K \quad (3.35)$$

with

$$g(\mathbf{z}) := \sum_i w_i \hat{X}_i \cdot e^{\sum_j a_{ij} z_j} . \quad (3.36)$$

From

$$\nabla_{\mathbf{z}} \left(\frac{1}{2} \cdot \mathbf{z}^\top \cdot \mathbf{z} - \lambda \cdot [g(\mathbf{z}) - K] \right) \Big|_{\mathbf{z}=\mathbf{z}^*} = 0 \quad (3.37)$$

we obtain

$$z_i^* - \lambda \cdot \sum_l a_{li} w_l \hat{X}_l \cdot e^{\sum_j a_{lj} z_j^*} = 0 \quad (3.38)$$

Equations (3.35) and (3.38) can be solved for λ and the elements of the vector \mathbf{z}^* by the aid of a nonlinear root finding algorithm such as NL2SOL [DGW81]. The effective geodesic strikes are then

$$K_i^* = \hat{X}_i \cdot e^{\xi_i} \quad (3.39)$$

with

$$\boldsymbol{\xi} = \mathbf{A} \cdot \mathbf{z}^* . \quad (3.40)$$

As for an initial guess for \mathbf{z}^* in equations (3.35) and (3.38), we can either use zero, or proceed to find an expansion as follows.

Defining

$$\bar{X} := \sum_i w_i \hat{X}_i \quad \text{and} \quad \kappa := \ln(K/\bar{X}) , \quad (3.41)$$

we rewrite (3.35) as

$$\sum_i w_i \hat{X}_i \cdot e^{\sum_j a_{ij} z_j^*} = \bar{X} \cdot e^\kappa . \quad (3.42)$$

We now seek an expansion given by

$$\lambda = 0 + \kappa \cdot \lambda^{(1)} + \kappa^2 \cdot \lambda^{(2)} + \dots \quad (3.43)$$

$$\mathbf{z}^* = 0 + \kappa \cdot \mathbf{z}^{(1)} + \kappa^2 \cdot \mathbf{z}^{(2)} + \dots \quad (3.44)$$

Substituting (3.44) into (3.42) and expanding in κ around 0 gives

$$\begin{aligned} \sum_i \omega_i \cdot \left[1 + \sum_j a_{ij} \left(\kappa z_j^{(1)} + \kappa^2 z_j^{(2)} \right) \right. \\ \left. + \frac{1}{2} \left(\sum_j a_{ij} \kappa z_j^{(1)} \right)^2 \right] \\ = 1 + \kappa + \frac{1}{2} \kappa^2 + \mathcal{O}(\kappa^3) \end{aligned} \quad (3.45)$$

where we have used the normalized effective weights

$$\omega_i := \frac{w_i \hat{X}_i}{\bar{X}}. \quad (3.46)$$

Matching coefficients up to order $\mathcal{O}(\kappa^2)$, equation (3.45) gives

$$\sum_i \omega_i = 1 \quad (3.47)$$

$$\boldsymbol{\omega}^\top \cdot \mathbf{A} \cdot \mathbf{z}^{(1)} = 1 \quad (3.48)$$

$$\boldsymbol{\omega}^\top \cdot \mathbf{A} \cdot \mathbf{z}^{(2)} = \frac{1}{2} - \frac{1}{2} \mathbf{z}^{(1)\top} \cdot \mathbf{A}^\top \cdot \boldsymbol{\Omega} \cdot \mathbf{A} \cdot \mathbf{z}^{(1)}, \quad (3.49)$$

with the matrix $\boldsymbol{\Omega}$ defined to be diagonal and its elements being equal to those of $\boldsymbol{\omega}$. Expanding (3.38) in κ yields

$$\begin{aligned} \kappa \cdot z_i^{(1)} + \kappa^2 \cdot z_i^{(2)} \\ - \left(\kappa \cdot \lambda^{(1)} + \kappa^2 \cdot \lambda^{(2)} \right) \cdot \sum_l w_l \hat{X}_l a_{li} \cdot \left[1 + \sum_j a_{lj} \kappa z_j^{(1)} \right] \\ = \mathcal{O}(\kappa^3) \end{aligned} \quad (3.50)$$

from which we derive

$$\mathbf{z}^{(1)} = \lambda^{(1)} \cdot \bar{X} \cdot \mathbf{A}^\top \cdot \boldsymbol{\omega} \quad (3.51)$$

$$\mathbf{z}^{(2)} = \lambda^{(1)} \cdot \bar{X} \cdot \mathbf{A}^\top \cdot \boldsymbol{\Omega} \cdot \mathbf{A} \cdot \mathbf{z}^{(1)} + \lambda^{(2)} \cdot \bar{X} \cdot \mathbf{A}^\top \cdot \boldsymbol{\omega} \quad (3.52)$$

by matching coefficients of κ up to second order. Combining (3.48) and (3.51) results in

$$\lambda^{(1)} = \frac{1}{\bar{X}} \cdot \frac{1}{\boldsymbol{\omega}^\top \cdot \mathbf{C} \cdot \boldsymbol{\omega}} \quad (3.53)$$

and hence

$$\mathbf{z}^{(1)} = \frac{\mathbf{A}^\top \cdot \boldsymbol{\omega}}{\boldsymbol{\omega}^\top \cdot \mathbf{C} \cdot \boldsymbol{\omega}}. \quad (3.54)$$

Further, combining (3.49) and (3.52) produces

$$\lambda^{(2)} = \frac{1}{\bar{X}} \cdot \frac{1}{\boldsymbol{\omega}^\top \cdot \mathbf{C} \cdot \boldsymbol{\omega}} \cdot \left[\frac{1}{2} - \frac{3}{2} \cdot \frac{\boldsymbol{\omega}^\top \cdot \mathbf{C} \cdot \boldsymbol{\Omega} \cdot \mathbf{C} \cdot \boldsymbol{\omega}}{(\boldsymbol{\omega}^\top \cdot \mathbf{C} \cdot \boldsymbol{\omega})^2} \right] \quad (3.55)$$

whence

$$\begin{aligned} \mathbf{z}^{(2)} = \frac{1}{2} \left(\mathbb{1} + 2 \cdot \frac{\mathbf{A}^\top \cdot \boldsymbol{\Omega} \cdot \mathbf{A}}{\boldsymbol{\omega}^\top \cdot \mathbf{C} \cdot \boldsymbol{\omega}} - 3 \cdot \frac{\boldsymbol{\omega}^\top \cdot \mathbf{C} \cdot \boldsymbol{\Omega} \cdot \mathbf{C} \cdot \boldsymbol{\omega}}{(\boldsymbol{\omega}^\top \cdot \mathbf{C} \cdot \boldsymbol{\omega})^2} \cdot \mathbb{1} \right) \\ \cdot \frac{\mathbf{A}^\top \cdot \boldsymbol{\omega}}{\boldsymbol{\omega}^\top \cdot \mathbf{C} \cdot \boldsymbol{\omega}} \end{aligned} \quad (3.56)$$

with $\mathbb{1}$ denoting the identity matrix. This finally gives us the second order expansion

$$\begin{aligned} \mathbf{z}^* = \kappa \cdot \frac{\mathbf{A}^\top \cdot \boldsymbol{\omega}}{\boldsymbol{\omega}^\top \cdot \mathbf{C} \cdot \boldsymbol{\omega}} \\ + \kappa^2 \cdot \frac{1}{2} \left(\mathbb{1} + 2 \cdot \frac{\mathbf{A}^\top \cdot \boldsymbol{\Omega} \cdot \mathbf{A}}{\boldsymbol{\omega}^\top \cdot \mathbf{C} \cdot \boldsymbol{\omega}} - 3 \cdot \frac{\boldsymbol{\omega}^\top \cdot \mathbf{C} \cdot \boldsymbol{\Omega} \cdot \mathbf{C} \cdot \boldsymbol{\omega}}{(\boldsymbol{\omega}^\top \cdot \mathbf{C} \cdot \boldsymbol{\omega})^2} \cdot \mathbb{1} \right) \\ \cdot \frac{\mathbf{A}^\top \cdot \boldsymbol{\omega}}{\boldsymbol{\omega}^\top \cdot \mathbf{C} \cdot \boldsymbol{\omega}} + \mathcal{O}(\kappa^3). \end{aligned} \quad (3.57)$$

As an example, we show in figure 3 the first and second order expansion solutions for a monthly observed Asian option of one year maturity in comparison with a numerical solution for an arbitrarily chosen term structure of arbitrage-free implied volatility.

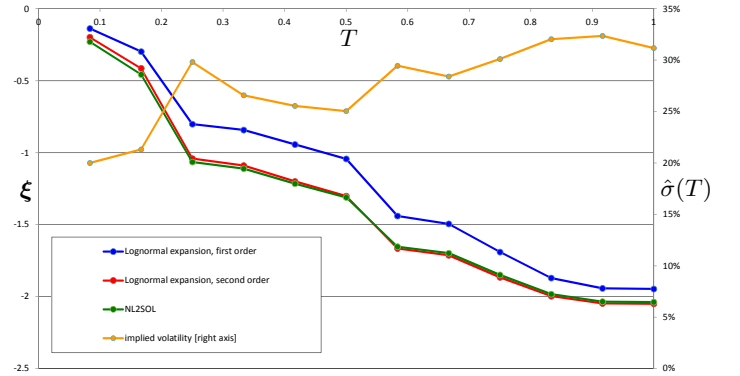


Figure 3: An example for the effective strike adjustments (3.40) of first and second order as given in (3.57) in comparison with a numerical solution by the aid of NL2SOL of equations (3.35) and (3.38) for a monthly observed Asian option of one year maturity with an arbitrarily chosen term structure of arbitrage-free implied volatility. All forwards were equal to 1, and the strike was at 0.3.

When any of the weights are negative, it is possible for \bar{X} to be zero. As a consequence, $\ln(K/\bar{X})$ is undefined, and none of the above equations can be evaluated. In that case, it may be better to use an expansion in

$$\tilde{\kappa} := K - \bar{X}, \quad (3.58)$$

which gives us, to first order

$$\tilde{\mathbf{z}}^* = \frac{\tilde{\kappa}}{\bar{X}} \cdot \frac{\mathbf{A}^\top \cdot \boldsymbol{\omega}}{\boldsymbol{\omega}^\top \cdot \mathbf{C} \cdot \boldsymbol{\omega}}$$

i.e.,

$$\tilde{z}_i^* = \frac{K - \bar{X}}{v_{\bar{X}}} \sum_j a_{ij} w_j \hat{X}_j \quad (3.59)$$

with

$$v_{\bar{x}} := \sum_{i,j} w_i \hat{X}_i c_{ij} w_j \hat{X}_j. \quad (3.60)$$

Analytical sensitivities of the exact solution

The geodesic strikes are obtained as the solution of equations (3.35) and (3.38). Multiplying (3.38) by A from the left, we reformulate this as

$$\begin{aligned} \xi - \lambda \cdot C \cdot \gamma &= 0 \\ \sum_j \gamma_j - K &= 0 \end{aligned} \quad (3.61)$$

with

$$\gamma_i := w_i \hat{X}_i e^{\xi_i} \equiv w_i K_i^* \quad (3.62)$$

for convenience of notation and brevity of the results. In order to compute the sensitivity of the geodesic strikes K_i^* to any input, which, for now, we indicate with a generic prime, i.e., $K_i^{* \prime}$, we use (3.39) giving us

$$K_i^{* \prime} = \frac{K_i^*}{\hat{X}_i} \cdot \hat{X}_i' + K_i^* \cdot \xi_i' \quad (3.63)$$

and from hereon focus on the computation of ξ_i' . Since the nonlinear system (3.61), which implicitly defines the geodesic strikes, is invariant to the change of any of the input parameters, we obviously have

$$\begin{aligned} [\xi - \lambda \cdot C \cdot \gamma]' &= 0 \\ [\sum_j \gamma_j - K]' &= 0. \end{aligned} \quad (3.64)$$

We combine this with the definitions

$$\Gamma := \text{diag}\{\gamma\} \quad (3.65)$$

$$\chi_i = \ln(\hat{X}_i) \quad (3.66)$$

and

$$\gamma_i' = \gamma_i \cdot \chi_i' + \gamma_i \cdot \xi_i' \quad (3.67)$$

into the generic sensitivity equation system

$$\begin{aligned} [\mathbb{1} - \lambda \cdot C \cdot \Gamma] \cdot \xi' - C \cdot \gamma \cdot \lambda' \\ = \lambda \cdot [C' \cdot \gamma + C \cdot \Gamma \cdot \chi'] \\ \gamma^\top \cdot \xi' = -\gamma^\top \cdot \chi' \end{aligned} \quad (3.68)$$

which is of course linear in the sensitivities ξ' and λ' . We note that this linear system has the same matrix of coefficients on the left hand side for all possible sensitivities that we might intend to compute: irrespective of whether we are interested in the sensitivity to a volatility, a correlation, or a forward, the left hand side is always the same, and it is always a linear system of size $m + 1$ if m is the number of geodesic strikes. It can thus be computationally advantageous to solve all sensitivities in one sweep, for instance with a single QR factorization or a single Moore-Penrose pseudo-inverse to safeguard against singular (under-determined) dependencies.

Sensitivity to forwards

In this case, the linear system (3.68) reduces to

$$\begin{aligned} [\mathbb{1} - \lambda \cdot C \cdot \Gamma] \cdot \xi^{(i)} - C \cdot \gamma \cdot \lambda^{(i)} &= \lambda \cdot C \cdot \Theta \cdot e_i \\ \gamma^\top \cdot \xi^{(i)} &= -e_i^\top \cdot \Theta \cdot e_i \end{aligned} \quad (3.69)$$

wherein e_i is the i -th unit basis vector,

$$\Theta := \text{diag}\{\gamma\} / \text{diag}\{\hat{X}\}, \quad (3.70)$$

and the superscript $(\cdot)^{(i)}$ represents the derivative with respect to the forward \hat{X}_i .

Sensitivity to the covariance matrix

Using the superscript notation $(\cdot)^{(kl)}$ for the derivative of any quantity (\cdot) with respect to the (k, l) -element c_{kl} of the covariance matrix C , we obtain

$$\begin{aligned} [\mathbb{1} - \lambda \cdot C \cdot \Gamma] \cdot \xi^{(kl)} - C \cdot \gamma \cdot \lambda^{(kl)} &= \lambda \cdot C^{(kl)} \cdot \gamma \\ \gamma^\top \cdot \xi^{(kl)} &= 0 \end{aligned} \quad (3.71)$$

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