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More likely than not

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Abstract

The *likelihood ratio* method [BG96] provides a tool for the calculation of hedge ratios and risk parameters with Monte Carlo calculations. In this article, we present the extension of existing likelihood ratio methods to multiple underlyings and to higher order sensitivities such as the cross-derivatives on the underlyings, and how the method can be used in a displaced diffusion [Rub83] framework. Next, an alternative likelihood ratio formula that can be used to compute sensitivities with respect to market adjustments of forward contracts is given. We also derive the respective first and second order hedge quantities with respect to the underlyings' hazard rates for the correlated-time-of-default model [Li00], as well as the corresponding correlation coefficient risk. In addition, we show how the sensitivities with respect to model parameters can be converted into quantities that are more meaningful to traders: exposures with respect to implied volatilities, or with respect to credit spreads. In a nutshell: this is a collection of likelihood ratio formulæ.

1 Introduction

The holy grail of numerical derivatives pricing is to find a fully generic method for the computation of the risk-neutral value of fully generic derivatives structures with little effort, *and to obtain all the possibly relevant hedge ratio figures along with the price*. Whilst the need for hedge ratios is given in all business areas for the purpose of dynamic replication and risk management, derivatives traders in certain asset classes such as FX options go one step further: for many exotic structures, in fact for most commoditised types of contracts, both parties of a deal not only agree on the price but also on the delta hedge that would neutralise the position with respect to movements in the underlying FX rate (to first order). This is done despite the fact that, strictly speaking, the used hedge ratios are effectively the main strategic decisions made by the exotic derivative trader, with or without the support of exotic pricing models. Naturally, the ability of a derivatives pricing utility to compute hedge ratios as well as prices is also desirable where the spot sensitivity hedge is not explicitly part of the actual deal done. There are several methods to compute the Greeks with Monte Carlo: brute-force finite differencing by means of revaluation with changed parameters, pathwise differentiation [Cur98, Jäc02], equivalent entropy projection methods [Ave98, AG02], fully fledged Malliavin calculus [FLL$^+$99] and of course the leaner cousin of Malliavin calculus: the likelihood ratio method [BG96].

1.1 A brief review of the likelihood ratio method

Option pricing by Monte Carlo simulation amounts to a numerical approximation to an integral

$$v = \int \pi(S)\ \psi(S)\,\mathrm{d}S \tag{1}$$

where $\pi(S)$ is the (numéraire-denominated, e.g. discounted) payoff as a function of a future realisation of a given set of underlying indices or assets denoted by S. Numerically, we construct evolutions of the underlying assets represented by S given a risk-neutral distribution density $\psi(S)$. We hereby typically construct the paths by the aid of a set of standard normal variates which corresponds to

$$v = \int \pi(S(z;\alpha))\ \varphi(z)\,\mathrm{d}z\ , \tag{2}$$

with $\varphi(z) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}z^2}$, and all dependencies on further pricing parameters (herein represented by α) such as the spot level at inception, volatility, time to maturity, etc., are absorbed into the path construction $S(z;\alpha)$. Any derivative with respect to any of the parameters will thus suffer from discontinuities of π in S (or discontinuities of derivatives of π with respect to S) since

$$\frac{\partial v}{\partial \alpha} = \int \frac{\partial}{\partial \alpha}\pi(S(z;\alpha))\ \varphi(z)\,\mathrm{d}z\ . \tag{3}$$

Clearly, discontinuities of π (or its derivatives with respect to S or any of the pricing or model parameters in the vector α) are commonplace in financial derivatives.

An intuitive way to understand the *likelihood ratio* method [BG96] is to change our view that the numéraire-denominated payoff π depends on S which in turn depends on the integration variable z, and instead to go back to the original equation (1) wherein not S but $\psi = \psi(S;\alpha)$ is effectively a function of the parameters α, albeit that the functional dependence may be somewhat more complicated. In other words, a transformation of the density is required to look at the pricing problem in the form

$$\frac{\partial v}{\partial \alpha} = \int \pi(S)\ \frac{\partial}{\partial \alpha}\psi(S;\alpha)\,\mathrm{d}S = \int \pi(S)\ \frac{\frac{\partial \psi(S;\alpha)}{\partial \alpha}}{\psi(S;\alpha)}\,\psi(S;\alpha)\,\mathrm{d}S\ . \tag{4}$$

The calculation of the desired Greek now looks exactly like the original pricing problem, only with a new payoff function

$$\chi(S;\alpha)\ :=\ \pi(S)\cdot\omega(S;\alpha) \tag{5}$$

with

$$\omega(S;\alpha)\ :=\ \frac{\frac{\partial \psi(S;\alpha)}{\partial \alpha}}{\psi(S;\alpha)}\ . \tag{6}$$

The term $\omega(S;\alpha)$ is often referred to as a *likelihood ratio*, whence the name of the method. Using this definition, the Greek calculation becomes

$$\frac{\partial v}{\partial \alpha} = \int \chi(S;\alpha)\,\psi(S;\alpha)\,\mathrm{d}S \tag{7}$$

with $\chi(S;\alpha)$ defined as in equation (5), i.e. each realised payoff in the simulation is simply weighted with the likelihood ratio $\omega(S;\alpha)$ in order to compute the desired Greek.

The beauty of this idea is that for the probability density functions that we typically use such as the one corresponding to (geometric) Brownian motion, the function $\chi(S;\alpha)$ is $\in \mathcal{C}^\infty$ in the parameter α and thus doesn't cause the trouble that we have when we approach the Greek calculation problem in the form of equation (2) by, say, finite differencing. The application is now straightforward. Alongside the calculation of the option price, for each constructed path (in whatever representation) in S, apart from computing the payoff, also calculate the likelihood ratio $\omega(S;\alpha)$, and multiply it with the payoff $\pi(S)$.

2 Transforming the joint density

Define $\boldsymbol{X}$ as a vector representing the evolution of the d state variables over all of a given set of m time steps $t \in \{t_1, ..., t_m\}$. A d-dimensional process for the state variables thus means that $\boldsymbol{X} \in \mathbb{R}^n$ with $n = d \cdot m$. Equally, define $\boldsymbol{z} \in \mathbb{R}^n$ as a vector of *uncorrelated* standard normal variates. In the following, we assume that the discretisation of the specific stochastic process at hand will be sampled, ultimately, by drawing standard normal variates and thus we base all decompositions on a vector of uncorrelated standard normal variates because ultimately all number generation (pseudo-random or low-discrepancy) starts off with vectors of normal variates that are uncorrelated in the limit of many draws. Of particular importance for all subsequent likelihood ratio method calculations is the global covariance matrix C of the realisation of all state variables at all future time horizons, i.e.

$$C = \left\langle \boldsymbol{X} \cdot \boldsymbol{X}^\top \right\rangle - \left\langle \boldsymbol{X} \right\rangle \cdot \left\langle \boldsymbol{X}^\top \right\rangle \ , \tag{8}$$

where angle brackets denote expectation in the chosen risk-neutral measure. Also, we need to define a pseudo-square root A of C such that[1]

$$C = A \cdot A^\top \ . \tag{9}$$

Note that A can be computed by the aid of a spectral split, a Brownian bridge decomposition, or by Cholesky decomposition (which corresponds to incremental path construction) [RJ00, Jäc02]. Since each draw $\boldsymbol{X}$ (representing a full evolution of all state variables over all future time horizons of interest) is fully determined by an underlying vector draw $\boldsymbol{z}$ of standard normal variates, we can view this as a transformation:

$$\boldsymbol{z} \ \rightarrow \ \boldsymbol{X} = \boldsymbol{X}(\boldsymbol{z}) \ . \tag{10}$$

Clearly, the function $\boldsymbol{X} = \boldsymbol{X}(\boldsymbol{z})$ depends on further parameters given by any one specific pricing problem at hand. In fact, the Greeks that we wish to calculate alongside our Monte Carlo simulation to determine the risk-neutral value of a deal are the partial derivatives of the contract with respect to these additional parameters that enter the transformation (10). The joint multivariate probability density describing the statistics of all of the elements of the vector $\boldsymbol{z}$ is given by

$$\phi(\boldsymbol{z}) := \prod_{i=1}^{n} \varphi(z_i) \tag{11}$$

with

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} \, \mathrm{e}^{-\frac{1}{2}z^2} \tag{12}$$

as before. By virtue of the transformation (10), this can be translated into a joint density function for $\boldsymbol{X}$

$$\psi(\boldsymbol{X}) = \phi(\boldsymbol{z}) \left| \frac{\partial(\boldsymbol{X})}{\partial(\boldsymbol{z})} \right|^{-1} \ , \tag{13}$$

provided that the Jacobian

$$J = \left(\frac{\partial(\boldsymbol{X})}{\partial(\boldsymbol{z})} \right) \tag{14}$$

is regular.

[1] Karatzas and Shreve call this matrix A the *dispersion matrix* [KS91] (page 284).

3 Multivariate geometric Brownian motion

For many derivatives that depend on a multitude of underlying assets, multi-factor geometric Brownian motion models are still an acceptable approach, especially when very little information on market implied skews and smiles of the individual underlyings is available. In this context, we define

$$x_{ji} := \ln S_{ji} = \ln S_i(t_j) \tag{15}$$

to represent the realisation of the i-th state variable at time t_j. Now, let us imagine that all Monte Carlo paths are created in an incremental fashion[2], i.e.

$$\begin{aligned} x_{ji} &= x_{(j-1)\,i} + \mu_{ji}\Delta t_j - \frac{1}{2}c_{jii} + \sum_{k=1}^{d} a_{jik} z_{jk} && (16) \\ &= x_{0i} + \sum_{l=1}^{j}\left(\mu_{li}\Delta t_l - \frac{1}{2}c_{lii} + \sum_{k=1}^{d} a_{lik} z_{lk}\right) . && (17) \end{aligned}$$

Hereby, all of the z_{jk} for $j = 1..m$, $k = 1..d$, are independent standard normal variates, and the a_{jik} comprise the elements of the pseudo-square root of the covariance matrix C_j for the time step $\Delta t_j = t_j - t_{j-1}$, as defined in equation (9). Equation (17) enables us to compute the elements of the Jacobian matrix J as defined in equation (14):

$$\frac{\partial x_{ji}}{\partial z_{lk}} = a_{lik}\mathbf{1}_{\{l\le j\}} \tag{18}$$

Note that this means that all of the elements of the Jacobian matrix are constants, and therefore any derivative of the Jacobian determinant with respect to either coordinate system ($\boldsymbol{x}$ or $\boldsymbol{z}$) vanish. What's more, equation (18) clearly states that the Jacobian matrix is a function of the covariance matrix for the time step, and *not* a function of the initial values of the state variables.

3.1 Sensitivities with respect to underlying assets

In order to compute the Greeks with respect to the initial values of the underlying assets, let us first recall that, using the definition in equation (15),

$$\frac{\partial}{\partial S_{0i}} = \frac{1}{S_{0i}}\frac{\partial}{\partial x_{0i}} \tag{19}$$

and that

$$\frac{\partial}{\partial x_{0i}}\psi(\boldsymbol{x}) \;=\; |J|^{-1}\cdot\frac{\partial}{\partial x_{0i}}\phi(\boldsymbol{z}) \;-\; \phi(\boldsymbol{z})\cdot|J|^{-2}\cdot\frac{\partial}{\partial x_{0i}}|J| \;. \tag{20}$$

Fortunately, the second term on the right hand side is identically zero by virtue of equation (18), and we can concentrate on the term $\frac{\partial}{\partial x_{0i}}\phi(\boldsymbol{Z})$. From the definitions (11) and (12), we can readily see

$$\frac{\partial}{\partial x_{0i}}\phi(\boldsymbol{z}) \;=\; -\phi(\boldsymbol{z})\cdot\sum_{j=1,k=1}^{j=m,k=d} z_{jk}\,\frac{\partial z_{jk}}{\partial x_{0i}} \tag{21}$$

[2] In section 3.5, we will generalise to arbitrary path decomposition.

whence

$$\frac{\partial}{\partial x_{0l}}\psi(\boldsymbol{x}) \;=\; -\psi(\boldsymbol{x}) \cdot \sum_{j=1,k=1}^{j=m,k=d} z_{jk}\,\frac{\partial z_{jk}}{\partial x_{0l}}\;. \tag{22}$$

In order to calculate the terms $\partial z_{jk}/\partial x_{0l}$, we take the partial differential of equation (16) with respect to x_{0l}, bearing in mind that the index j in that equation is an integer in the range $[1, m]$. This gives

$$0 \;=\; \delta_{j1}\delta_{il} + \sum_{k=1}^{d} a_{jik}\frac{\partial z_{jk}}{\partial x_{0l}} \tag{23}$$

where δ represents the Kronecker symbol. Let us now define the inverse of the dispersion matrix A_j for time step $t_{j-1} \rightarrow t_j$ as

$$B_j := A_j^{-1}\;. \tag{24}$$

This enables us to write

$$\frac{\partial z_{jk}}{\partial x_{0l}} \;=\; -\sum_{i=1}^{d} b_{jki}\delta_{j1}\delta_{il} \;=\; -b_{jkl}\delta_{j1}\;, \tag{25}$$

and

$$\frac{\partial}{\partial x_{0l}}\psi(\boldsymbol{x}) \;=\; -\psi(\boldsymbol{x}) \cdot \sum_{k=1}^{d} z_{1k}b_{1kl}\;. \tag{26}$$

Combining this with equations (6) and (19), we finally obtain the likelihood ratio for the computation of the sensitivity with respect to the initial value of asset #i:

$$\underline{\underline{\omega_{\Delta_i} \;=\; \sum_{k=1}^{d} \frac{z_{1k}b_{1ki}}{S_{0i}}}} \tag{27}$$

Note that the only non-constant elements on the right hand side are the normal variates that are used for the first step in the incremental path construction, despite the fact that the derivative contract may contain an arbitrary payoff schedule along the path. This is a consequence of the telescopic (i.e. additive) nature of equation (17): any change of any underlying along the path could have been caused by movements in the first step.

As for the second derivatives with respect to the initial values of the underlying assets, we proceed as follows:

$$\omega_{\Gamma_{ij}} \;=\; \frac{1}{\psi}\partial_{S_{0j}}\left(\psi \cdot \omega_{\Delta_i}\right) \;=\; \omega_{\Delta_i}\omega_{\Delta_j} + \partial_{S_{0j}}\left(\omega_{\Delta_i}\right) \tag{28}$$

which leads to

$$\underline{\underline{\omega_{\Gamma_{ij}} \;=\; \omega_{\Delta_i}\omega_{\Delta_j} - \delta_{ij}\frac{\omega_{\Delta_i}}{S_{0j}} - \sum_{k=1}^{d}\frac{b_{1ki}b_{1kj}}{S_{0i}S_{0j}}}}\;. \tag{29}$$

Note that the last term is constant and can be pre-calculated. What's more, the expression $\sum_{k=1}^{d} b_{1ki}b_{1kj}$ is equal to the entry in the i-th row and j-th column of the inverse of the covariance matrix over the first time step. This can be seen as follows:

$$\sum_{k=1}^{d} b_{1ki}b_{1kj} = \sum_{k=1}^{d} (B_1)_{ki}(B_1)_{kj} = \sum_{k=1}^{d}\left(A_1^{-1\,\top}\right)_{ik}\left(A_1^{-1}\right)_{kj} = \left(\left(A_1^{\top}\right)^{-1}\cdot A^{-1}\right)_{ij} = \left(\left(A\cdot A^{\top}\right)^{-1}\right)_{ij} = \left(C_1^{-1}\right)_{ij}\;. \tag{30}$$

A major caveat with respect to the application of the likelihood ratio method is the requirement of non-singular covariance matrices. This, however, is perfectly consistent with the overall model description of correlated Brownian motion. Let us recall that the hedge ratios Delta and Gamma are sensitivities with respect to the changes of the values of underlying assets *as they can occur dynamically over the course of small time steps dt within the model framework*. For instance, the hedge ratio Delta is computed as the number of underlying shares that have to be held against a short position in a derivative contract in order to protect against losses in the hedged portfolio due to dynamic movements of the underlying asset values, in the specified model framework. Any individual Delta figure represents the number of shares that have to be held in order to protect against movements of the associated asset alone, *ceteris paribus*, i.e. whilst all other market parameters and asset values remain unchanged. This is naturally impossible if we assume a singular correlation matrix to start with. As an example, think of two assets with perfect correlation. It is *by definition of perfect correlation* impossible for one of the assets to move alone whilst the other remains unchanged. Therefore, we cannot expect the likelihood ratio method to return sensible Delta and Gamma figures when the provided correlation matrices are singular. *The likelihood ratio method provides Delta and Gamma sensitivity numbers with respect to dynamic changes of the underlying securities.*

3.2 Forward vega

In order to compute a measure for the sensitivity of an exotic option price with respect to the individual coefficients representing *volatility*, let us first assume that any payoff function is a function of a vector $\boldsymbol{x}$ of transformed state variables, and *not* a function of the volatility coefficients, neither directly, nor indirectly. In a Gaussian Hull-White or extended Vasicek model, this precludes the possibility of the payoff function requiring additional transformations from $\boldsymbol{x}$ to a set of discount bond values invoking functionals of the short rate volatility function(s). However, for interest rate models that are based on the dynamics of market observables such as the BGM/J [BGM97, Jam97] and other market models [MSS97], or in the case of correlated geometric Brownian motion or displaced (geometric) diffusion [Rub83], this restriction poses no difficulty [GZ99].

As a second assumption, let us take it as given that all volatilities are considered to be constant over the time step $t_{j-1} \rightarrow t_j$. This can be interpreted as either the assumption of piecewise constant instantaneous volatility, or as the computed exposure to be the sensitivity with respect to the root-mean-square volatility over the time step. Let σ_{ji} denote the volatility coefficient of the underlying financial variable $\#i$ over time step $t_{j-1} \rightarrow t_j$. For the purpose of Vega calculations, we thus need to establish the volatility likelihood ratio

$$\omega_{\sigma_{ji}} \;:=\; \frac{\frac{\partial \psi(\boldsymbol{x})}{\partial \sigma_{ji}}}{\psi(\boldsymbol{x})} \;. \tag{31}$$

Now let us recall that we express the density $\psi(\boldsymbol{x})$ explicitly as a product of the joint density of independent normal variates which are given by the inverse transformation $\boldsymbol{z} = \boldsymbol{z}(\boldsymbol{x};\boldsymbol{\sigma})$, and a Jacobian determinant, i.e.

$$\psi(\boldsymbol{x}) = |J|^{-1} \cdot \phi\big(\boldsymbol{z}\left(\boldsymbol{x};\boldsymbol{\sigma}\right)\big) \;. \tag{32}$$

For the calculation of Delta and Gamma, there was no need to compute derivatives of the Jacobian determinant with respect to the spot value of the underlying financial variable because the Jacobian determinant does not depend on it. For Vega, however, the situation is different. The density $\psi(\boldsymbol{x})$

depends on the volatility coefficients both directly in the Jacobian determinant *and* indirectly due to the inverse transformation $\boldsymbol{x} \to \boldsymbol{z} = \boldsymbol{z}(\boldsymbol{x}; \boldsymbol{\sigma})$. This means,

$$\frac{\partial}{\partial \sigma_{ji}} \psi(\boldsymbol{x}) \;=\; \phi\big(\boldsymbol{z}\,(\boldsymbol{x}; \boldsymbol{\sigma})\,\big) \cdot \frac{\partial}{\partial \sigma_{ji}} \, |J|^{-1} \;+\; |J|^{-1} \cdot \frac{\partial}{\partial \sigma_{ji}} \phi\big(\boldsymbol{z}\,(\boldsymbol{x}; \boldsymbol{\sigma})\,\big) \;\; . \tag{33}$$

In equation (18), we obtained the result that the Jacobian matrix J as defined in equation (14) is of block-triangular form. Therefore, the Jacobian determinant $|J|$ is given by the product of the Jacobian determinants of the block matrices that form the diagonal part of J, namely by the product of the determinants of the stepwise dispersion matrices, that is

$$|J| \;=\; \prod_{j=1}^{m} |A_j| \;\; . \tag{34}$$

At this point, it is conducive to make a specific choice as to how the covariance matrix C_j for time step $t_{j-1} \to t_j$ is split into the pseudo-square root A_j. For this purpose, define Θ_j as the diagonal matrix whose diagonal entries are given by the volatility coefficients, i.e.

$$\theta_{jik} \;:=\; \sigma_{ji} \delta_{ik} \;\; . \tag{35}$$

Also, let R_j be the correlation matrix for the time step $t_{j-1} \to t_j$. The covariance matrix can thus be written as

$$C_j \;=\; \Theta_j \cdot R_j \cdot \Theta_j \cdot \Delta t_j \;\; . \tag{36}$$

In order to obtain a pseudo-square root A_j of C_j, let us further define the matrix Q_j given by the product of the spectral pseudo-square root of the correlation matrix, and the square root of the time step Δt_j, i.e.

$$Q_j \;:=\; \sqrt{R_j} \cdot \sqrt{\Delta t_j} \tag{37}$$

and define

$$A_j \;:=\; \Theta_j \cdot Q_j \;\; . \tag{38}$$

Given this formulation of the dispersion matrix A_j, it is clear that we have

$$|A_j| \;=\; |Q_j| \cdot \prod_{i=1}^{d} \sigma_{ji} \tag{39}$$

and

$$\frac{\partial}{\partial \sigma_{kl}} \, |A_j|^{-1} \;=\; -\frac{\delta_{jk}}{\sigma_{jl}} \cdot |A_j|^{-1} \;\; . \tag{40}$$

This enables us to express the first term on the right hand side of equation (33) in the following simple and convenient way:

$$\phi\big(\boldsymbol{z}\,(\boldsymbol{x}; \boldsymbol{\sigma})\,\big) \cdot \frac{\partial}{\partial \sigma_{ji}} \, |J|^{-1} \;=\; -\frac{1}{\sigma_{ji}} \phi\big(\boldsymbol{z}\,(\boldsymbol{x}; \boldsymbol{\sigma})\,\big) \cdot |J|^{-1} \;=\; -\frac{1}{\sigma_{ji}} \psi(\boldsymbol{x}) \tag{41}$$

The second term of equation (33) can be expanded in analogy to equation (21), and thus we have

$$\frac{\partial}{\partial \sigma_{ji}} \psi(\boldsymbol{x}) \;=\; \Psi(\boldsymbol{x}) \cdot \left[-\frac{1}{\sigma_{ji}} - \sum_{k=1,l=1}^{k=m,l=d} z_{kl} \, \frac{\partial z_{kl}}{\partial \sigma_{ji}} \right] \;\; . \tag{42}$$

In order to compute the term $\frac{\partial z_{kl}}{\partial \sigma_{ji}}$, it is best to rewrite equation (16) as

$$x_{kn} \;=\; x_{(k-1)\,n} + \mu_{kn}\Delta t_k - \frac{1}{2}\sigma_{kn}^2\Delta t_k + \sum_{l=1}^{d} a_{knl} z_{kl} \;. \tag{43}$$

Differentiating equation (43) with respect to σ_{ji} yields

$$0 \;=\; -\sigma_{ji}\Delta t_j\,\delta_{jk}\delta_{in} + \sum_{l=1}^{d} z_{kl}\frac{\partial a_{knl}}{\partial \sigma_{ji}} + \sum_{l=1}^{d} a_{knl}\frac{\partial z_{kl}}{\partial \sigma_{ji}} \;. \tag{44}$$

By virtue of equation (38), we have

$$\frac{\partial a_{knl}}{\partial \sigma_{ji}} \;=\; \frac{1}{\sigma_{ji}} a_{jil}\delta_{jk}\delta_{in} \tag{45}$$

whence equation (44) becomes

$$\sum_{l=1}^{d} a_{knl}\frac{\partial z_{kl}}{\partial \sigma_{ji}} \;=\; \left(\sigma_{ji}\Delta t_j - \sum_{g=1}^{d}\frac{1}{\sigma_{ji}} a_{jig} z_{jg}\right)\delta_{jk}\delta_{in} \;. \tag{46}$$

Using the inverse $B_k = A_k^{-1}$, this equation can be solved:

$$\begin{aligned}\frac{\partial z_{kl}}{\partial \sigma_{ji}} \;&=\; \sum_{n=1}^{d} b_{kln}\left(\sigma_{ji}\Delta t_j - \sum_{g=1}^{d}\frac{1}{\sigma_{ji}} a_{jig} z_{jg}\right)\delta_{jk}\delta_{in} \\ &=\; b_{jli}\left(\sigma_{ji}\Delta t_j - \sum_{g=1}^{d}\frac{1}{\sigma_{ji}} a_{jig} z_{jg}\right)\delta_{jk} \;.\end{aligned} \tag{47}$$

Substituting this into equation (42) gives us

$$\frac{\partial}{\partial \sigma_{ji}}\psi(\boldsymbol{x}) \;=\; \Psi(\boldsymbol{x})\cdot\left[\sum_{l=1}^{d} z_{jl} b_{jli}\left(\sum_{k=1}^{d}\frac{1}{\sigma_{ji}} a_{jik} z_{jk} - \sigma_{ji}\Delta t_j\right) - \frac{1}{\sigma_{ji}}\right] \;. \tag{48}$$

Eventually, we obtain the following expression for the likelihood ratio for Vega:

$$\omega_{\sigma_{ji}} \;=\; \frac{1}{\sigma_{ji}}\left(\sum_{k=1}^{d} z_{jk} b_{jki}\right)\left(\sum_{l=1}^{d} a_{jil} z_{jl}\right) - \sigma_{ji}\Delta t_j\left(\sum_{k=1}^{d} z_{jk} b_{jki}\right) - \frac{1}{\sigma_{ji}} \tag{49}$$

3.3 Projection onto vega with respect to implied volatilities

The likelihood ration given by equation (49) enables us to compute the sensitivity of an exotic derivative contract with respect to forward volatility coefficients. For hedging purposes, however, we are usually more interested in sensitivities with respect to the implied volatilities of plain vanilla options available in the market. This requires a transformation from $\omega_{\sigma_{ji}}$ to $\omega_{\hat{\sigma}_{ji}}$ where the hat on the volatility coefficient is to indicate that it is an implied volatility of plain vanilla options on the underlying financial asset #i with maturity t_j. The details for this transformation can be found in [Jäc02] (section 11.9.1), and the resulting relationship is

$$\omega_{\hat{\sigma}_{ji}} \;=\; \left(\frac{\hat{\sigma}_{ji} t_j}{\sigma_j \Delta t_j}\right)\omega_{\sigma_{ji}} - \mathbf{1}_{\{j<m\}}\left(\frac{\hat{\sigma}_{ji} t_j}{\sigma_{j+1\,i}\Delta t_{j+1}}\right)\omega_{\sigma_{j+1\,i}} \;. \tag{50}$$

3.4 Displaced diffusion corrections

The likelihood ratio formulæ derived in the previous sections are directly linked to the model assumption of (correlated) geometric Brownian motion. It is possible, however, to adapt these formulæ to closely related modelling assumptions. One such example is that of *displaced diffusion* [Rub83]. In this case, the dynamics of any one individual underlying asset value S are given by

$$\frac{\mathrm{d}Y}{Y} \;=\; \mu\,\mathrm{d}t + \sigma_{\text{DD}}\mathrm{d}W \tag{51}$$

with

$$Y(t) \;:=\; S(t) + A(t) \tag{52}$$

and $A(t)$ being a time-dependent displacement whose evolution over time is determined by

$$A(t) \;=\; A_0\mathrm{e}^{\mu t} \quad \text{and} \quad A_0 \;:=\; -S_0\log_2 Q \tag{53}$$

for some displacement coefficient $Q \in (0,2)$. Given that the displaced diffusion model can be seen as that of geometric Brownian motion of an affine transformation of the underlying asset, it is clear that

$$\omega_{\Delta^{\text{DD}}_{S_0}} \;=\; \omega_{\Delta^{\text{DD}}_{Y_0}} \cdot \frac{\partial Y_0}{\partial S_0}\,. \tag{54}$$

Since, as mentioned at the end of section 3.1, likelihood ratio Delta and Gamma sensitivity calculations are with respect to dynamic changes of the underlying asset, we have to view Y_0 as

$$Y_0 \;=\; S_0 + A_0 \tag{55}$$

with A_0 *not* being directly or indirectly dependent on S_0. Therefore, the term $\partial Y_0/\partial S_0$ in equation (54) is unity and we have

$$\underline{\underline{\omega^{\text{DD}}_{\Delta_{S_0}} \;=\; \omega^{\text{DD}}_{\Delta_{Y_0}}}} \tag{56}$$

and

$$\underline{\underline{\omega^{\text{DD}}_{\Gamma_{S_0}} \;=\; \omega^{\text{DD}}_{\Gamma_{Y_0}}}}\,. \tag{57}$$

As for Vega, the situation is slightly different in the displaced diffusion setting. This is mainly due to the fact that when we talk about Vega, we really are interested in a sensitivity with respect to the implied Black volatility of plain vanilla options on the underlying assets since it is those

plain vanilla options that we would use for hedging purposes. Of course, using the volatility coefficient likelihood ratio given in equation (49) and the transformation (50), we can directly compute the sensitivity $\partial\psi_{\text{DD}}/\partial\hat{\sigma}_{\text{DD}}$ with respect to the displaced diffusion coefficient $\hat{\sigma}_{\text{DD}}$. The sensitivity with respect to the at-the-forward implied Black volatility is then given by

$$\frac{\partial\psi_{\text{DD}}}{\partial\hat{\sigma}_{\text{Black}}} \;=\; \frac{\partial\psi_{\text{DD}}}{\partial\hat{\sigma}_{\text{DD}}} \cdot \frac{\partial\hat{\sigma}_{\text{DD}}}{\partial\hat{\sigma}_{\text{Black}}}\,. \tag{58}$$

Define a simplified form of the (undiscounted) Black (call) option formula by

$$\mathrm{B}(F,K,\zeta) \;:=\; F\cdot\mathrm{N}\left(\frac{\ln\left({}^{F}\!/_{K}\right)}{\zeta}+\frac{1}{2}\zeta\right) - K\cdot\mathrm{N}\left(\frac{\ln\left({}^{F}\!/_{K}\right)}{\zeta}-\frac{1}{2}\zeta\right) \tag{59}$$

with F being the par forward price for maturity T, K the strike of the option, and $\zeta = \hat{\sigma}_{\text{Black}}\sqrt{T}$ the log-standard deviation of the log-normally distributed forward price.

Now, the implied Black volatility that corresponds to the at-the-forward price of a plain vanilla option valued in the displaced diffusion setting is implicitly determined by

$$\text{B}\left(F + A(T), F + A(T), \hat{\sigma}_{\text{DD}}\sqrt{T}\right) \;=\; \text{B}\left(F, F, \hat{\sigma}_{\text{Black}}\sqrt{T}\right) \tag{60}$$

whence we have

$$\left(\left.\frac{\partial \text{B}(f,k,\zeta)}{\partial \zeta}\right|_{\substack{f=F+A(T)\\k=K+A(T)\\\zeta=\hat{\sigma}_{\text{DD}}\sqrt{T}}}\right)\cdot\frac{\partial\hat{\sigma}_{\text{DD}}}{\partial\hat{\sigma}_{\text{Black}}} \quad=\quad \left(\left.\frac{\partial \text{B}(f,k,\zeta)}{\partial \zeta}\right|_{\substack{f=F\\k=K\\\zeta=\hat{\sigma}_{\text{Black}}\sqrt{T}}}\right) \tag{61}$$

Defining

$$\vartheta \;:=\; \frac{F + A(T)}{F} \tag{62}$$

and taking advantage of the specific structure of the Black option formula, we can put all of this together to obtain

$$\omega^{\text{DD}}_{\hat{\sigma}\text{Black}} \quad=\quad \omega^{\text{DD}}_{\hat{\sigma}\text{DD}}\;\cdot\;\left(\left.\frac{\partial \text{B}(f,k,\zeta)}{\partial \zeta}\right|_{\substack{f=1\\k=1\\\zeta=\hat{\sigma}_{\text{Black}}\sqrt{T}}}\right)\Bigg/\left(\left.\frac{\partial \text{B}(f,k,\zeta)}{\partial \zeta}\right|_{\substack{f=\vartheta\\k=\vartheta\\\zeta=\hat{\sigma}_{\text{DD}}\sqrt{T}}}\right)\quad. \tag{63}$$

3.5 Global covariance path decomposition

In practice, we may prefer to construct paths of correlated geometric Brownian motion with methods other than the straightforward incremental algorithm assumed in equations (16) and (17). Instead, we may wish to use a spectral decomposition of the global covariance matrix, or a combination of the Brownian bridge with spectral decomposition of the correlation between assets. In any case, we can always assume that we have knowledge of all the entries c_{ij} of a global covariance matrix C that determines the covariance of the realisation of any of the d assets at any of the m points in time to any other of the the assets at any other time. In order to reduce the number of indices, let us define

$$n \;:= m \cdot d \tag{64}$$

and use the convention that with $i = k \cdot d + l$, the variable

$$x_i \;=\; \ln S_{kl} \tag{65}$$

refers to the realisation of asset #l at time step t_k. Equally, with $j = p \cdot d + q$, let the covariance matrix entry c_{ij} denote the covariance of asset #l at time step t_k with asset #q at time step t_p. Also, let us define

$$\xi_i \;=\; \ln F_{kl} - \frac{1}{2}c_{ii} \tag{66}$$

where F_{kl} stands for the par strike of a forward contract on asset #l maturing at time t_k. Using this enumeration scheme, a vector draw $\boldsymbol{z} \in \mathbb{R}^n$ tranforms into a realisation of the state variable vector

$$\boldsymbol{x} = \boldsymbol{\xi} + A \cdot \boldsymbol{z}\;. \tag{67}$$

Hereby, A represents the dispersion matrix, i.e. the pseudo-square root of the covariance matrix C that corresponds to the chosen path construction method. For incremental generation, the matrix A is of triangular form; for a Brownian bridge method, it is sparse with some structure; and for the spectral path construction method, it is a full matrix. The density of the state vector $\boldsymbol{x}$, or rather its mean-corrected equivalent

$$\boldsymbol{y} := \boldsymbol{x} - \boldsymbol{\xi} \ , \tag{68}$$

is given by

$$\psi(\boldsymbol{y}) \ = \ \frac{\mathrm{e}^{-\frac{1}{2}\,\boldsymbol{y}^\top D\,\boldsymbol{y}}}{|A| \cdot (2\pi)^{n/2}} \tag{69}$$

with

$$D \ := \ C^{-1} \ . \tag{70}$$

Following a similar analysis to the previous sections, we can compute the likelihood ratio for the sensitivity with respect to the forward value F_i (assuming the same enumeration scheme as for the x_i in order to reduce the number of indices) as

$$\begin{aligned} \omega_{F_i} \ &= \ \partial_{F_i}\psi / \, \psi \\ &= \ \partial_{F_i} y_i \cdot \partial_{y_i} \left(-\frac{1}{2}\, \boldsymbol{y}^\top D\, \boldsymbol{y} \right) \ , \end{aligned}$$

i.e.

$$\underline{\underline{\omega_{F_i} \ = \ \frac{1}{F_i} \cdot \sum_{j=1}^{n} d_{ij} y_j \ .}} \tag{71}$$

Similarly, the second order sensitivities, or *forward cross-Gammas*, turn out to be

$$\underline{\underline{\omega_{F_i F_j} \ = \ \omega_{F_i}\omega_{F_j} - \delta_{ij}\frac{\omega_{F_i}}{F_j} - \frac{d_{ij}}{F_i F_j} \ .}} \tag{72}$$

The reader should note, however, that the above likelihood ratios are with respect to the movement of a *single* forward contract par strike. This means, if we wish to compute the spot delta, we need to use the conversion

$$\omega_{\Delta_l} \ = \ \sum_{k=1}^{m} \omega_{F_{(k\cdot d+l)}} \frac{\partial F_{(k\cdot d+l)}}{\partial S_{0l}} \ . \tag{73}$$

The conversion formula for spot gamma clearly simply involves a double sum, respectively. The above decomposition into forward deltas (and gammas) can be useful for two reasons. The first one is that, in case the time between the valuation date and the first fixing date is short, the variance of likelihood ratio term for sensitivity with respect to the first forward grows like $(\Delta t_1)^{-1/2}$ and can spoil the whole calculation. A remedy for this can be to approximate the sensitivity with respect to the first fixing with other means, and then to sum up the terms to obtain a spot delta. The second benefit of forward delta calculations is that, for instance for equity underlyings, it allows us to correct for the specific bespoke model for the connection between the spot, the forward, and any expected dividends between now and the fixing date we wish to employ[3]. This can be particularly pertinent for some of the single stock names that, relative to their current share prices, are expected to pay considerable dividends in the near future, which can give rise to noticeable pricing and hedge ratio differences.

[3]The most commonly used formula for the forward is to subtract the net present value of dividend forecasts, but other approaches such as to treat them as a mixture of absolute and proportional components are also in use.

For the computation of the likelihood ratio associated with the sensitivity with respect to any of the implied volatilities $\hat{\sigma}_i$, we use again the fact that the covariance matrix can be represented as the product

$$C \;=\; \Theta \cdot R \cdot \Theta \tag{74}$$

with the entries of the diagonal matrix Θ given by

$$\theta_{ii} \;=\; \hat{\sigma}_i \sqrt{t_{i\%d}} \tag{75}$$

and R being the global correlation matrix. The notation $t_{i\%d}$ is to indicate that the index i is actually given by the $i = k \cdot d + l$ with l being the asset index and k being the time step index, i.e. $k = i\%d$. This gives us immediately

$$\begin{aligned} \partial_{\hat{\sigma}_i} |A|^{-1} &= \partial_{\hat{\sigma}_i} \left(\sqrt{|C|} \right)^{-1} \\ &= \partial_{\hat{\sigma}_i} \left(\sqrt{|R| \cdot \prod_{i=1}^{n} \hat{\sigma}_i^2 t_{i\%d}} \right)^{-1} \\ &= -1 \,/ (\hat{\sigma}_i \cdot |A|) \;\; , \end{aligned} \tag{76}$$

$$\partial_{\hat{\sigma}_i} y_j \;=\; \tfrac{c_{ii}}{\hat{\sigma}_i} \delta_{ij} \;, \tag{77}$$

$$\partial_{\hat{\sigma}_i} d_{jk} \;=\; -\tfrac{d_{jk}}{\hat{\sigma}_i} \left(\delta_{ij} + \delta_{ik} \right) \;, \tag{78}$$

and

$$\omega_{\hat{\sigma}_i} \;=\; \partial_{\hat{\sigma}_i} \psi / \, \psi \;=\; -\tfrac{1}{\hat{\sigma}_i} - \tfrac{1}{2} \partial_{\hat{\sigma}_i} \big(\boldsymbol{y}^\top D \, \boldsymbol{y} \big) \tag{79}$$

which eventually leads to

$$\underline{\underline{\omega_{\hat{\sigma}_i} \;=\; \frac{1}{\hat{\sigma}_i} \left[(y_i - c_{ii}) \sum_{j=1}^{n} d_{ij} y_j \;-\; 1 \right] \;.}} \tag{80}$$

4 Likelihood ratio risk in a correlated-time-of-default model

A popular approach to value credit derivatives such as *Collateralised Debt Obligations* is to use a model whose sole function is to govern the joint distribution of default times of different credit entities. When default times are drawn from the joint distribution by means of first generating codependent uniform variates coupled via a Gaussian copula, and then using the individual marginal distributions associated with the individual underlyings, the framework is often also referred to as a *correlated-time-of-default model*, and sometimes as the *Li* model [Li00]. This modelling approach is very convenient due to the comparative ease of implementation and the resulting speed when used with Monte-Carlo simulations, as well as the availability of analytical solutions when the correlation structure is restricted to specific forms [HW03].

In a model that is reduced to the distribution of default times of the n underlyings that are part of the credit derivative contract, the value of the deal can be written as

$$v = \int \pi(\boldsymbol{\tau}) \cdot \psi(\boldsymbol{\tau}) \; \mathrm{d}\tau^n \tag{81}$$

where $\pi(\boldsymbol{\tau})$ is the discounted payoff as a function of a future realisation of the n default times τ_i for $i = 1..n$, and $\psi(\boldsymbol{\tau})$ denotes the joint probability density of the default times. The joint distribution of the default times is in turn given as

$$\Psi(\boldsymbol{\tau}) = C(\boldsymbol{u}) \tag{82}$$

where $C(\boldsymbol{u})$ is a copula as a function of the default quantiles $\boldsymbol{u}$, and the marginal distributions of the default times are given by $\Psi_i(\tau_i)$ for $i = 1..n$. The connection between the default quantiles, or alternatively the survival probabilities $Q_i := 1 - u_i$, and the associated default times, is what essentially constitutes the default model, and we will be more specific on this later. For now, suffice it to say that we assume the existence of the inverse function $Q_i^{-1}(\cdot)$ which enables us to infer a default time from a given drawn uniform default quantile u_i, i.e.

$$\tau_i = Q_i^{-1}(1 - u_i) \ . \tag{83}$$

In the case of a Gaussian copula the joint density becomes

$$\psi(\boldsymbol{\tau}) = \varphi(\boldsymbol{y}; R) \cdot \left| \frac{\partial(\boldsymbol{y})}{\partial(\boldsymbol{\tau})} \right| \qquad \text{with} \qquad \varphi(\boldsymbol{y}; R) = \frac{1}{\sqrt{(2\pi)^n \, |R|}} \cdot \mathrm{e}^{-\frac{1}{2}\boldsymbol{y}^T R^{-1} \boldsymbol{y}} \tag{84}$$

where R represents the matrix of Gaussian correlation coefficients, and the connection between the Gaussian variates and the default time draws is given by the cumulative normal and survival probability function:

$$\Phi(y_i) \ = \ 1 - Q_i(\tau_i) \tag{85}$$

Since linear correlation coefficients between default times are very difficult to compare and estimate from market observable prices or time series, it is generally advisable to use a *rank correlation* coefficient instead. One such rank correlation measure is known as *Spearman's rho* and is nothing other than the linear correlation computed from the marginal quantile variables $\boldsymbol{u}$. Luckily, there is a very close relationship between the Gaussian correlation number $\varrho_{y_i y_j}$ and the quantile correlation $\rho_{u_i u_j}$, namely

$$\rho_{u_i u_j} \ = \ \frac{6}{\pi} \cdot \arcsin\left(\frac{1}{2} \cdot \varrho_{y_i y_j}\right) \qquad \text{or equivalently} \qquad \varrho_{y_i y_j} \ = \ 2 \cdot \sin\left(\frac{\pi}{6} \cdot \rho_{u_i u_j}\right) \ . \tag{86}$$

The relationship is almost, albeit not entirely, linear, as shown in figure 1.

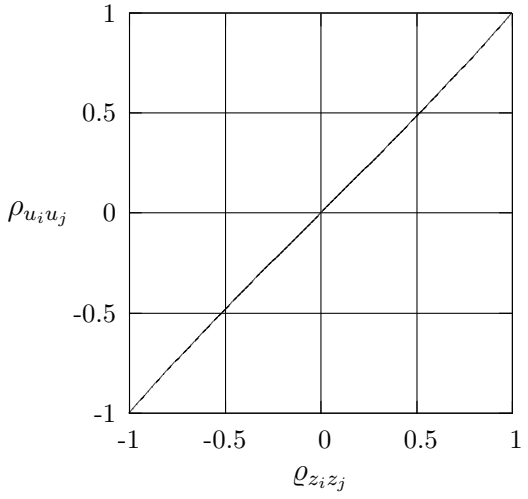

Figure 1: Spearman's rho of correlated Gaussian variates. Not quite a straight line, but nearly.

4.1 Hazard Rate Risk

The most common approach to associate potential default times with a survival probability is to use the concept of an instantaneous forward hazard rate curve $h(t)$:

$$Q_i(\tau_i) \;:=\; \mathrm{e}^{-H_i(\tau_i)} \qquad \text{with} \qquad H_i(\tau_i) \;:=\; \int_0^{\tau_i} h_i(s)\,\mathrm{d}s \tag{87}$$

The exact form of the sensitivity of the value of a CDO to changes in the hazard rate curve $h_i(t)$ of the i^{th} underlying will depend on how that curve is constructed. In any implementation, in the most general sense, a *curve* can be defined by an interpolation rule and a set of parameters specific to the interpolation rule. Of course, the interpolation rule may be given by a fully parametric description of the curve depending on an arbitrary parameter vector $\boldsymbol{\lambda}_i$, or indeed, by two vectors of associated numbers, namely abscissa and ordinate values, and a specifically chosen interpolation algorithm such as piecewise constant (either left- or right-continuous in the interpolation points), piecewise linear, natural splines, monotone cubic, or otherwise. Either way, the functional form of the hazard rate curve will be of the form

$$h_i(t) = h_i(t;\boldsymbol{\lambda}_i)$$

wherein $\boldsymbol{\lambda}_i$ represents the vector of parameters that could change whenever the credit default swap rates of the associated underlying credit index vary. In the following, we will assume that the hazard rate curve is defined by a vector of abscissa-ordinate pairs, and an interpolation rule. In other words λ_{kl} refers to the hazard rate of credit number k that prevails precisely at time t_l, i.e.

$$\lambda_{kl} := h_k(t_l), \tag{88}$$

given an arbitrary time discretisation of the hazard rate curve $h_k(t)$ over the times $t_0, t_1, \ldots, t_m$, with $t_0 := 0$, and a chosen interpolation rule.

Our first objective is now to derive the likelihood ratio required for the calculation of the parametric sensitivity of any given credit derivative with respect to λ_{kl},

$$\omega_{\lambda_{kl}} := \frac{\partial_{\lambda_{kl}}\psi(\boldsymbol{\tau})}{\psi(\boldsymbol{\tau})}, \tag{89}$$

wherein ψ represents, as before, the joint probability density of the specifically drawn vector of default times, and $\partial_{\lambda_{kl}}$ stands for the partial derivative $\partial/\partial\lambda_{kl}$. For this, we need to compute

$$\left|\frac{\partial(\boldsymbol{y})}{\partial(\boldsymbol{\tau})}\right| \;=\; \prod_{i=1}^{n}\frac{h_i Q_i}{\varphi_i} \tag{90}$$

which follows directly from equation (85) by explicit calculation and wherein we have dropped the explicit mentioning of the dependencies, i.e. $h_i = h_i(\tau_i)$, $Q_i = Q_i(\tau_i)$, and $\varphi_i = \varphi(y_i)$. Combining equations (84) and (90), we obtain

$$\omega_{\lambda_{kl}} = y_k\partial_{\lambda_{kl}}y_i - \sum_{j=1}^{n} y_j\tilde{\varrho}_{jk}\partial_{\lambda_{kl}}y_k + \frac{\partial_{\lambda_{kl}}h_k}{h_k} - \partial_{\lambda_{kl}}H_k \tag{91}$$

where we have defined

$$\tilde{\varrho}_{ij} := \left(R^{-1}\right)_{ij}\;. \tag{92}$$

From (85), we derive

$$\partial_{\lambda_{kl}} y_i = \delta_{ik} \frac{Q_i}{\varphi_i} \partial_{\lambda_{il}} H_i \tag{93}$$

which gives us

$$\omega_{\lambda_{kl}} = \frac{Q_k y_k}{\varphi_k} \partial_{\lambda_{kl}} H_k - \frac{Q_k \sum_{j=1}^{n} \tilde{\varrho}_{kj} y_j}{\varphi_k} \partial_{\lambda_{kl}} H_k + \frac{\partial_{\lambda_{kl}} h_k}{h_k} - \partial_{\lambda_{kl}} H_k \tag{94}$$

The first order hazard rate risk formula (89) finally becomes[4]

$$\underline{\underline{\omega_{\lambda_{kl}} = \frac{\partial_{\lambda_{kl}} h_k}{h_k} + \left[\frac{Q_k}{\varphi_k} \cdot (y_k - \tilde{y}_k) - 1 \right] \partial_{\lambda_{kl}} H_k}} \tag{95}$$

with $H_k(\tau_k)$ and $Q_k(\tau_k)$ defined as in equation (87), and the variable $\tilde{y}_k$ standing for the k-th entry of the solution vector $\tilde{\boldsymbol{y}}$ of the linear system

$$R \cdot \tilde{\boldsymbol{y}} = \boldsymbol{y} \ . \tag{96}$$

The generic formulation of the second order sensitivity with respect to the hazard rate node levels is given by[5]

$$\omega_{\lambda_{kl}\lambda_{pq}} \ = \ \frac{\partial_{\lambda_{kl}\lambda_{pq}} \psi}{\psi} \tag{97}$$

and by straightforward differentiation rules, we can re-express this as

$$\omega_{\lambda_{kl}\lambda_{pq}} \ = \ \omega_{\lambda_{kl}} \omega_{\lambda_{pq}} + \partial_{\lambda_{pq}} \omega_{\lambda_{kl}} \ . \tag{98}$$

The unknown term $\partial_{\lambda_{pq}} \omega_{\lambda_{kl}}$ on the right hand side can be readily computed:

$$\begin{aligned} \partial_{\lambda_{pq}} \omega_{\lambda_{kl}} \ = \ & \frac{1}{h_k^2} \left[h_k \partial_{\lambda_{kl}\lambda_{pq}} h_k - \partial_{\lambda_{kl}} h_k \cdot \partial_{\lambda_{pq}} h_k \right] + \left[\frac{Q_k}{\varphi_k} \cdot (y_k - \tilde{y}_k) - 1 \right] \partial_{\lambda_{kl}\lambda_{pq}} H_k \\ & + \left[\left(\partial_{\lambda_{pq}} Q_k + Q_k \cdot y_k \cdot \partial_{\lambda_{pq}} y_k \right) (y_k - \tilde{y}_k) + Q_k \cdot \left(\partial_{\lambda_{pq}} y_k - \partial_{\lambda_{pq}} \tilde{y}_k \right) \right] \frac{\partial_{\lambda_{kl}} H_k}{\varphi_k} \ . \end{aligned} \tag{99}$$

The term $\partial_{\lambda_{pq}} \tilde{y}_k$ is hereby, not surprisingly, given by the k-th entry of the solution vector $\partial_{\lambda_{pq}} \tilde{\boldsymbol{y}}$ to the linear problem

$$R \cdot \partial_{\lambda_{pq}} \tilde{\boldsymbol{y}} = \partial_{\lambda_{pq}} \boldsymbol{y} \ . \tag{100}$$

Note that, from the fact that hazard rate curves of different credit entities are assumed to be independent, and from equation (85), we have

$$\partial_{\lambda_{pq}} y_k = \delta_{kp} \cdot \frac{Q_k}{\varphi_k} \cdot \partial_{\lambda_{kq}} H_k \ . \tag{101}$$

In the case of (left-continuous) piecewise constant interpolation, using the auxiliary definitions

$$\gamma_{kl} \ := \ \partial_{\lambda_{kl}} h_k(\tau_k; \boldsymbol{\lambda}) \ = \ \mathbf{1}_{\{\tau_k \in (t_{l-1}, t_l]\}} \tag{102}$$

and

$$\kappa_{kl} \ := \ \partial_{\lambda_{kl}} H_k(\tau_k; \boldsymbol{\lambda}) \ = \ \mathbf{1}_{\{\tau_k > t_{l-1}\}} \cdot (\tau_k - t_{l-1}) \tag{103}$$

[4] It is the author's pleasure to point out that the first derivation of the presented credit default swap rate delta, albeit in different form and notation, was done by his former colleagues Mark Seaborne and Rhodri Wynne.

[5] The author is grateful to his former colleague Sanjeev Shukla for the implementation of the second order hazard rate likelihood risk equations, and for the many numerical experiments he conducted to confirm their validatity.

equations (95) and (99) can be simplified to

$$\omega_{\lambda_{kl}} = \frac{\gamma_{kl}}{h_k} + \left[\frac{Q_k}{\varphi_k} \cdot (y_k - \tilde{y}_k) - 1\right] \kappa_{kl} \tag{104}$$

and

$$\partial_{\lambda_{pq}} \omega_{\lambda_{kl}} = \kappa_{kl}\kappa_{pq} \cdot \frac{Q_k}{\varphi_k \varphi_p} \left[\delta_{pk} \left((Q_p y_p - \varphi_p)(y_p - \tilde{y}_p) + Q_p\right) - Q_p \cdot \tilde{\varrho}_{pk}\right] - \frac{\delta_{pk}\delta_{ql}\gamma_{kl}}{h_k^2} \ . \tag{105}$$

For the even more special case that we are only interested in second order derivatives with respect to node levels of one and the same hazard rate curve, we have

$$\partial_{\lambda_{kq}} \omega_{\lambda_{kl}} = \kappa_{kl}\kappa_{kq} \cdot \frac{Q_k}{\varphi_k^2} \left[Q_k \cdot (1 - \tilde{\varrho}_{kk}) + (Q_k y_k - \varphi_k)(y_k - \tilde{y}_k)\right] - \frac{\delta_{ql}\gamma_{kl}}{h_k^2} \ . \tag{106}$$

4.2 Projection onto credit default swap rates

The risk figures that can be computed using the likelihood ratios derived in the previous sections are with respect to the hazard rate levels prevailing at the nodes that are specific to the used interpolation discretisation. In practice, we are usually more interested in the risk expressed as a sensitivity with respect to what practically amounts to the *time-averaged* hazard rates, i.e.

$$\hat{\lambda}_{kl} := \frac{1}{t_l} \int_0^{t_l} h_k(t) \, \mathrm{d}t \ . \tag{107}$$

For piecewise constant interpolation, we therefore have

$$t_l \cdot \hat{\lambda}_{kl} \ - \ t_{l-1} \cdot \hat{\lambda}_{k\,l-1} = \Delta t_l \cdot \lambda_{kl} \tag{108}$$

with $\Delta t_l := t_l - t_{l-1}$. This means, that given the sensitivities of the value of a derivative v with respect to the hazard rate node levels λ_{kl}, we have

$$\partial_{\hat{\lambda}_{kl}} v = \sum_{j=1}^{l} \frac{\Delta t_j}{t_l} \partial_{\lambda_{kj}} v \tag{109}$$

and

$$\partial_{\hat{\lambda}_{kl}\hat{\lambda}_{pq}} v = \sum_{j=1}^{l} \sum_{r=1}^{q} \frac{\Delta t_j}{t_l} \frac{\Delta t_r}{t_q} \partial_{\lambda_{kj}\lambda_{pr}} v \ . \tag{110}$$

4.3 Correlation risk[6]

It follows from equation (84) that, in order to obtain

$$\omega_{\rho_{ij}} \quad = \quad \frac{\partial_{\rho_{ij}} \psi}{\psi} \ , \tag{111}$$

[6]The author would like to thank his former colleague Sanjeev Shukla for helpful discussions on the analysis in this section.

we need to calculate

$$\frac{\partial_{\rho_{ij}}\psi}{\psi} \;=\; \frac{\partial_{\rho_{ij}}\varphi(\boldsymbol{y};R)}{\varphi(\boldsymbol{y};R)} \;=\; \partial_{\rho_{ij}}\ln\varphi(\boldsymbol{y};R) \;=\; -\frac{1}{2}\cdot\partial_{\rho_{ij}}\left(\boldsymbol{y}^\top\cdot R^{-1}\cdot\boldsymbol{y}+\ln|R|\right)\,. \tag{112}$$

Define $\mathbf{1}_{(ij)}$ as the matrix whose sole non-zero entry is the element at position (i,j) with value 1. Then, starting from

$$R\cdot R^{-1} \;=\; \mathbf{1}\,, \tag{113}$$

we have

$$\begin{aligned}
\partial_{\rho_{ij}}\left(R\cdot R^{-1}\right) &= 0\\
\left(\mathbf{1}_{(ij)}+\mathbf{1}_{(ji)}\right)\cdot R^{-1}+R\cdot\partial_{\rho_{ij}}\left(R\cdot^{-1}\right) &= 0\\
\partial_{\rho_{ij}}\left(R\cdot^{-1}\right) &= -R^{-1}\cdot\left(\mathbf{1}_{(ij)}+\mathbf{1}_{(ji)}\right)\cdot R^{-1}
\end{aligned}$$

The likelihood ratio required for the calculation of the sensitivity of the contract value with respect to one of the pairwise correlation coefficients ρ_{ij} for $i<j$ is therefore

$$\omega_{\rho_{ij}} \;=\; \frac{\partial_{\rho_{ij}}\psi}{\psi} \;=\; \frac{1}{2}\cdot\boldsymbol{y}^\top\cdot R^{-1}\cdot\left(\mathbf{1}_{(ij)}+\mathbf{1}_{(ji)}\right)\cdot R^{-1}\cdot\boldsymbol{y} \;-\; \frac{1}{2}\partial_{\rho_{ij}}\ln|R|\,. \tag{114}$$

Using the generic linear algebra result that

$$\partial_{m_{ij}}|M| \;=\; (-1)^{i+j}\cdot|M|\cdot\left(M^{-1}\right)_{ij}\,, \tag{115}$$

for any invertible matrix M, and that we only consider the upper right triangle of the correlation matrix to have independent entries, we have

$$\partial_{\rho_{ij}}|R| \;=\; 2\cdot(-1)^{i+j}\cdot|R|\cdot\left(R^{-1}\right)_{ij} \tag{116}$$

and thus obtain

$$\omega_{\rho_{ij}} \;=\; \frac{1}{2}\cdot\boldsymbol{y}^\top\cdot R^{-1}\cdot\left(\mathbf{1}_{(ij)}+\mathbf{1}_{(ji)}\right)\cdot R^{-1}\cdot\boldsymbol{y} \;-\; (-1)^{i+j}\cdot\left(R^{-1}\right)_{ij} \tag{117}$$

which reduces to

$$\underline{\underline{\omega_{\rho_{ij}} \;=\; \tilde{y}_i\tilde{y}_j \;-\; (-1)^{i+j}\cdot\tilde{\varrho}_{ij}}} \tag{118}$$

using the definitions (92) and (96).

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